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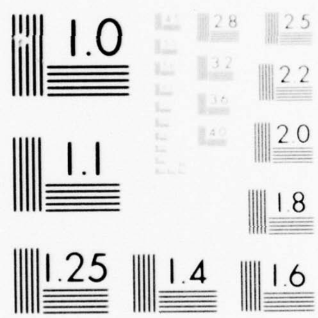

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Piecewise Continuous Solutions of Pseudoparabolic Equations in Two Space Dimensions †

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I. Introduction

One of the principal boundary value problems in analytic function theory is the so called RIEMANN boundary value problem. The simplest version of the problem requires the finding of an analytic function ϕ in $\mathbb{C} \setminus \Gamma$, where Γ is a closed smooth contour, and a prescribed Hölder continuous jump is prescribed for ϕ across Γ. The solution of this problem may be given in terms of a Cauchy integral (see [5], [8], and [10]). In generalized analytic, as well as generalized hyperanalytic function theory, a Cauchy-type representation exists, which suggest that the Riemann problem may be solved in a similar way. This problem was solved in [2]; whereas, in [1] the second major boundary value problem, that associated with Hilbert, was solved for generalized hyperanalytic function theory.

In [7] pseudoparabolic equations of the form

$$(1) \qquad \underset{\sim}{L}w := \frac{\partial}{\partial t}\left[w_{\bar z} + aw + b\bar{w}\right] + cw + d\bar{w} = 0 \ ,$$

where $a, b, c, d \in L_{p,2}(\mathbb{C})$, $2 < p$ were investigated. Integral representations reminiscent of those occurring for generalized analytic functions were found to be valid. This permits the posing and solving of what we refer to as an <u>initial-boundary value problem</u> of the Riemann type. The considerations of [7] concerning the hypercomplex operator

(2) $$\frac{\partial}{\partial t}\left[Dw + aw + b\overline{w}\right] + cw + d\overline{w} \quad ,$$

where

(3) $$Dw: = \frac{\partial w}{\partial \overline{z}} + q(z)\,\frac{\partial w}{\partial z}\ ,$$

with $$q(z): = \sum_{k=1}^{r-1} e^k q_k(x,y)\ ,$$

(Here e is a nilpotent with $e^r = 0$.) suggests a natural extension should be possible here. Indeed, similar results should also be available for the metaparabolic equations of the form,

(4) $$w_{\overline{z}} + aw + b\overline{w} + cw_t + d\overline{w}_t = 0\ ,$$

and

(5) $$Dw + aw + b\overline{w} + cw_t + d\overline{w}_t = 0\ .$$

In our work we make the simplifying assumption that the coefficients vanish identically in the unbounded component of $\mathbb{C}\setminus\Gamma$. This condition may be removed if sufficient decay is imposed on the coefficients as $z \to \infty$; however, this leads to further technical considerations. Moreover, we have only considered the case where Γ consists of a single contour. It would be of interest to treat the case where D^+ is multi-connected, but the multiplicity is a piecewise constant with respect to t .

II. Representations of Solutions

By means of the Pompeiu operator the pseudoparabolic equation

$$(1) \qquad \underset{\sim}{L}w = 0 \ , \quad \underset{\sim}{L}w := \frac{\partial}{\partial t}\left(w_{\bar{z}} + aw + b\bar{w}\right) + cw + d\bar{w} \ ,$$

$$a, b, c, d \in L_{p,2}(\mathbb{C}) \quad (2 < p)$$

may be reformulated as the integral equation

$$(2) \qquad w - \frac{1}{\pi}\int_{\mathbb{C}} (aw + b\bar{w}) \frac{d\xi d\eta}{\zeta - z} - \frac{1}{\pi}\int_{\mathbb{C}}\int_0^t (cw + d\bar{w})d\tau \frac{d\xi d\eta}{\zeta - z}$$

$$= w(z,0) - \frac{1}{\pi}\int_{\mathbb{C}} \left\{aw(\zeta,0) + b\overline{w(\zeta,0)}\,\Phi\frac{d\xi d\eta}{\zeta - z} + \Phi(z,t)\right\} ,$$

where $\Phi_{\bar{z}t}(z,t) \equiv 0$. Here

$$\Phi(z,t) = \sum_{k=0}^{\infty} a_k(t) z^k \qquad (z \in \mathbb{C},\ t \in \mathbb{R}),\ \Phi(z,0) \equiv 0 \ ,$$

and $\Phi(z,t)$ is a differentiable function of t .

If for each $t \in \mathbb{R}$, $w(z,t)$ is a bounded function in $\mathbb{C}$, then $\Phi(z,t)$ is a bounded analytic function in $\mathbb{C}$ for each t. By Lionville's Theorem $\Phi(z,t)$ is a function of t alone i.e. $\Phi(z,t) \equiv \Phi(t)$. In this case as $z \to \infty$ in (2) one has

$$\Phi(t) := w(\infty,t) - w(\infty,0) \ , \text{ i.e. } \ \Phi(0) = o \ ,$$

consequently for bounded solutions (1) is equivalent to

$$(3)\quad w - \frac{1}{\pi}\int_{\mathbb{C}} (aw+b\bar{w})\,\frac{d\xi d\eta}{\zeta-z} - \frac{1}{\pi}\int_{\mathbb{C}}\int_0^t (cw+d\bar{w})\,d\tau\,\frac{d\xi d\eta}{\zeta-z} = \Phi(t) + \phi(z),$$

where

$$(4)\qquad \phi(z) := w(z,0) - \frac{1}{\pi}\int_{\mathbb{C}} \left(aw(\zeta,0) + b\overline{w(\zeta,0)}\right)\frac{d\xi d\eta}{\zeta-z}\ .$$

If now w is a continuous solution of (3) bounded in $z \in \mathbb{C}$ for each $t \in R$ then

$$\Phi \in C(\mathbb{R})\ ,\quad \phi \in C(\mathbb{C})\ .$$

We consider now (3) for given data $\Phi \in C(\mathbb{R})$, $\phi \in C(\mathbb{C})$.

<u>Lemma 1</u>: <u>Let</u> $\Phi \in C(\mathbb{R})$, $\phi \in C(\mathbb{C})$, <u>and the coefficients</u> a, b <u>satisfy the inequality</u>

$$(5)\qquad \frac{1}{\pi}\int_{C}\left(|a(\zeta)| + |b(\zeta)|\right)\frac{d\xi d\eta}{|\zeta-z|} \leq \alpha < 1 \qquad (z \in \mathbb{C}).$$

Then in the space $B_{\mathbb{C}}(\mathbb{C}\times\mathbb{R})$ <u>of functions which are bounded in</u> $z \in \mathbb{C}$ <u>for each</u> $t \in \mathbb{R}$ <u>equation</u> (3) <u>is uniquely solvable</u>.

<u>Proof</u>: The difference $\omega := w_1 - w_2$ of two solutions of (3) having the same initial data, and asymptotic behavior as $z \to \infty$, is a solution of the homogeneous equation

$$\omega = \underset{\sim}{T}\omega := \frac{1}{\pi}\int_{\mathbb{C}} (a\omega+b\bar{\omega})\,\frac{d\xi d\eta}{\zeta-z} + \frac{1}{\pi}\int_{\mathbb{C}}\int_0^t (c\omega+d\bar{\omega})\,d\tau\,\frac{d\xi d\eta}{\zeta-z}\ .$$

If β denotes an upper bound of

$$\int_C \left(|c| + |d|\right) \frac{d\xi d\eta}{|\zeta - z|}$$

and

$$||\omega||_1 := \sup_{\substack{z\varepsilon\mathbb{C},\\ |t|\leq 1}} |\omega(z,t)|$$

then

$$||\underset{\sim}{T}\omega||_1 \leq \left(\alpha+\beta|t|\right)||\omega||_1 \ .$$

It follows that T is a contractive operator if

$$|t| < \mathrm{Min}\left\{1, \frac{1-\alpha}{\beta}\right\};$$

hence, $\omega = T\omega$ has only the trivial solution

$$\omega(z,t) \equiv 0 \quad \text{for} \quad z \in \mathbb{C} \ , \ |t| \leq t_o := \mathrm{Min}\left\{1, \frac{1-\alpha}{\beta}\right\} .$$

The statement holds also for $|t| = t_0$ because of continuity. As the equation (1) is an autonomous differential equation with respect to t we can extend this conclusion to read

$$\omega(z,t) \equiv 0 \ (z \in \mathbb{C}, \ t \in \mathbb{R}).$$

Let $B^p(\mathbb{C})$ now stand for the class of functions having real derivatives up to order p which are continuous and bounded.

<u>Corollary</u>: <u>Let</u> $w(\infty,t) \in C(\mathbb{R})$ <u>and</u> $w(z,0) \in B^o(\mathbb{C})$. <u>Furthermore</u>, <u>let the inequality</u> (5) <u>hold</u>. <u>Then equation</u> (1) <u>is uniquely solvable in</u> $B_{\mathbb{C}}(\mathbb{C} \times \mathbb{R})$.

Lemma 2: Let $\Phi \in C^1(\mathbb{R})$, $\phi \in B^1(\mathbb{C})$, and inequality (5) hold. Then integral equation (3) is solvable.

Proof. Equation (3) can be written in the form

$$w - \underset{\sim}{T}w = \Phi + \phi$$

and be solved by iteration. If we put

$$w_o := \Phi + \phi \ , \quad w_k = \Phi + \phi + \underset{\sim}{T}w_{k-1} \quad (k \in \mathbb{N})$$

then

$$w = \lim_{k\to\infty} w_k = \sum_{k=0}^{\infty} \underset{\sim}{T}^k(\Phi+\phi)$$

is a solution of (3). The convergence of the series follows from the estimates

$$|(\underset{\sim}{T}^k w)(z,t)| \leq \sum_{\ell=0}^{k} \binom{k}{\ell} \alpha^{k-\ell} \beta^{\ell} \frac{|t|^{\ell}}{\ell!} ||w_o||_t$$

$$\left(||w_o||_t = \sup_{\substack{z\in\mathbb{C}, \\ |\tau|\leq|t|}} |w_o(z,\tau)| \right)$$

and

$$|w(z,t)| \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\beta}{1-\alpha}\right)^k |t|^k \frac{||w_o||_t}{1-\alpha} \qquad (z \in \mathbb{C},\ t \in \mathbb{R}).$$

In the following we are interested in special "bounded" solutions of (1). If ϕ given by (4) is a bounded analytic function in $\mathbb{C}$ then ϕ must be a constant, i.e. $\phi(z) \equiv w(\infty,0)$ so that

$$\Phi(t) + \phi(z) \equiv w(\infty,t).$$

By this we are led to consider the equation

(6) $\quad w - \underset{\sim}{T}w = \psi,$

where ψ is a differentiable function of t in $\mathbb{R}$.

(i) ψ <u>is a real function</u>: The unique solution of (6) is given by

$$\sum_{k=0}^{\infty} T_{\psi}^{k} = \sum_{k=0}^{\infty} \alpha_k \psi_k$$

where α_k are functions of z alone and ψ_k are given by iterated integration of ψ

$$\psi_o : = \psi \qquad \psi_k(t) : = \int_0^t \psi_{k-1}(\tau)\, d\tau \qquad (k \in \mathbb{N}) .$$

The functions α_k are independent of ψ, and are determined by the functions a, b, c, d of the operator $\underset{\sim}{L}$; they fulfill the following integral recurrence formulae.

$$\alpha_o - \frac{1}{\pi} \int_{\mathbb{C}} \left(a\alpha_o + b\, \overline{\alpha_o} \right) \frac{d\xi d\eta}{\zeta - z} = 1 \ ,$$

(7)

$$\alpha_k - \frac{1}{\pi} \int_{\mathbb{C}} \left(a\alpha_k + b\, \overline{\alpha_k} \right) \frac{d\xi d\eta}{\zeta - z} = \frac{1}{\pi} \int_{\mathbb{C}} \left(c\alpha_{k-1} + d\overline{\alpha_{k-1}} \right) \frac{d\xi d\eta}{\zeta - z} \quad (k \in \mathbb{N}) .$$

This system is solvable (see Vekua [9]). α_o has to be the first function F of the generating pair (F,G) corresponding to the functions (a,b) (see Bers [3]) and the other α_k in the following, denoted by F_k, are then uniquely defined from F . **The solution of (6), for given real ψ , may be seen** to be given by

(8) $\quad \left(\underset{\sim}{F}\, \psi\right)(z,t) : = \sum_{k=0}^{\infty} F_k(z) \psi_k(t) \quad (z \in \mathbb{C},\ t \in \mathbb{R}) F_o : = F.$

A majorant of this series can be found by

$$\sup_{z\in\mathbb{C}} |F(z)| \sup_{|\tau|\leq|t|} |\psi(\tau)| \exp \frac{\beta}{1-\alpha}|t| .$$

(ii) $\underset{\sim}{\psi}$ <u>is a pure imaginary function</u>: Then as in (i) one obtains a solution of (6) in the form

$$(9)\qquad (\underset{\sim}{G}\ \psi)(z,t) := \sum_{k=0}^{\infty} G_k(z)\psi_k(t) \quad (z \in \mathbb{C},\ t \in \mathbb{R})$$

where $G_o := G$ is the second function of the generating pair (F,G) and the G_k are the solutions of the system

$$(10)\qquad \begin{aligned} &G_o - \frac{1}{\pi}\int_{\mathbb{C}} \left(aG_o + b\overline{G}_o\right) \frac{d\xi d\eta}{\zeta - z} = i \quad , \\ &G_k - \frac{1}{\pi}\int_{\mathbb{C}} \left(aG_k + b\overline{G}_k\right) \frac{d\xi d\eta}{\zeta - z} = \frac{1}{\pi}\int_{\mathbb{C}} \left(cG_{k-1} + d\overline{G_{k-1}}\right) \frac{d\xi d\eta}{\zeta - z} \quad (k \in \mathbb{N}). \end{aligned}$$

The coefficients F_k and G_k appearing in the representations for $\underset{\sim}{F}$, respectively $\underset{\sim}{G}$, have the additional properties

$$F_o(\infty) = 1,\ G_o(\infty) = i,\ F_k(\infty) = 0,\ G_k(\infty) = 0 \quad (k \in \mathbb{N}) .$$

(iii) $\underset{\sim}{\psi} = \psi_1 + i\psi_2$ <u>is a complex function</u>: Let w be the solution of (6) then

$$\tilde{w} := w - \underset{\sim}{F}\psi_1 - \underset{\sim}{G}\psi_2$$

is a solution of the homogeneous problem

$$\tilde{w} - \underset{\sim}{T}\tilde{w} = 0$$

as can be seen by

$$\left(\underset{\sim}{F}\psi_1 + \underset{\sim}{G}\psi_2\right)(z,0) = F_o(z)\psi_1(0) + G_o(z)\psi_2(0)$$

$$\left(\underset{\sim}{F}\psi_1 + \underset{\sim}{G}\psi_2\right)(\infty,t) = \psi_1(t) + i\psi_2(t) = \psi(t)$$

and the corresponding function ϕ given by (4)

$$\phi(z) \equiv \psi(0)$$

which follows by the first equations in (7) and (10). With this we conclude that $\tilde{w}$ is identically zero; hence, we have the representation

$$w = \underset{\sim}{F}\psi_1 + \underset{\sim}{G}\psi_2 .$$

Generally every solution of (1) with vanishing initial data can be represented by the resolvent of an integral equation when the coefficients a, b, c, d of $\underset{\sim}{L}$ are asked to vanish outside of the closure of a bounded domain D with rectifiable boundary Γ.

<u>Theorem 1</u>. <u>A solution of</u> (1) <u>in</u> $\hat{D} \times \mathbb{R}$, <u>having zero initial data</u> $w(z,0) \equiv 0$, <u>may be represented as</u>

$$(11)\quad \frac{1}{2\pi i}\int_0^t \int_\Gamma \Big[w_\tau(\zeta,\tau)\Omega_\tau^{(1)}(z,t;\zeta,\tau)d\zeta - \overline{w_\tau(\zeta,\tau)\Omega_\tau^{(2)}(z,t;\zeta,\tau)d\zeta}\Big]d\tau$$

$$= \begin{cases} w(z,t) \ , & z \in D \\ 0 \ , & z \notin \overline{D} \end{cases} , \quad t \in \mathbb{R} .$$

<u>The</u> $\Omega^{(k)}$ <u>are the fundamental kernels of</u> $\underset{\sim}{L}$ <u>and are given by</u>

(12a) $\Omega^{(k)}(z,t;\zeta,\tau) := X^{(1)}(z,t;\zeta,\tau) +$

$$(-1)^{k-1} i X^{(2)}(z,t;\zeta,\tau), \ (k=1,2).$$

Remark: Here the $X^k(z,t;\zeta,\tau)$ are a system of fundamental solutions of (1) having the Vekua $X^{(k)}(z,\zeta)$ fundamental solutions as the initial coefficients, namely

(12b) $$X^{(k)}(z,t;\zeta,\tau) = \sum_{\nu=0}^{\infty} X_\nu^{(k)}(z,\zeta) \frac{(t-\tau)^{\nu+1}}{(\nu+1)!} \quad (k=1,2) .$$

(See Gilbert-Schneider [7] and Vekua [9] pg. 167)

(12b) are convergent for all real t and τ, and

$$X_o^{(k)}(z,\zeta) = \frac{(-i)^{k-1}}{2(\zeta-z)} \exp\left[\omega^{(k)}(z)-\omega^{(k)}(\zeta)\right],$$

$$\omega^{(k)} \varepsilon B^\alpha(\mathbb{C}) \ , \ \omega^{(k)}(z) = 0(|z|^\alpha) \ (z \to \infty), \ \alpha = \frac{p-2}{2} .$$

The $X_\nu^{(k)}(z,\zeta)(\nu \varepsilon \mathbb{N}, k=1,2)$ are solutions of the system (7). As may be seen $X_1^{(k)}(z,\zeta) \varepsilon L_{p_1}(\hat{D}), 2<p_1$; hence, $X_2^{(k)}(z,\zeta)$ belongs to $C^\alpha(\hat{D})$ for each $\zeta \varepsilon \mathbb{C}$ and $X_\nu^{(k)}(z,\zeta) = 0(|z|^{-1})(z \to \infty, \zeta \varepsilon \mathbb{C}, \nu \varepsilon \mathbb{N}_o)$ (see Gilbert-Schneider [7]). An estimation of the series in (12b) shows

$$\left|X^{(k)}(z,t;\zeta,\tau) - X_0^{(k)}(z,\zeta) - \frac{1}{2} X_1^{(k)}(z,\zeta)(t-\tau)\right| \leq$$

$$\leq \left(\frac{1-\alpha}{\beta}\right)^3 \left[\exp \frac{\beta}{1-\alpha}|t-\tau| - \sum_{\nu=0}^{2} \left(\frac{\beta}{1-\alpha}\right)^\nu \frac{|t-\tau|^\nu}{\nu!}\right] \sup_{z \varepsilon D} \left|X_2^{(k)}(\zeta,z)\right|$$

for each $\zeta \ \varepsilon \ \mathbb{C}$, t and τ real .

$\Omega^{(k)}$ has the following local behavior

$$(13)\quad \begin{cases} \Omega^{(1)}(z,t;\zeta,\tau) - \dfrac{t-\tau}{\zeta - z} = 0(|\zeta - z|^{-\beta}|t-\tau|) \\ \qquad\qquad\qquad\qquad (\zeta - z \to 0, t-\tau \to 0, \beta = \frac{2}{p} < 1), \\ \Omega^{(2)}(z,t;\zeta,\tau) = 0(|\zeta - z|^{-\beta}|t-\tau|) \\ \Omega^{(k)}(z,t;\zeta,\tau) = 0(|z|^{-1}|t-\tau|) \quad (z \to \infty,\ k = 1,2). \end{cases}$$

The proof of theorem 1 is given in [7].

<u>Theorem 2</u>. <u>If</u> $w(z,t)$ <u>is analytic in</u> $\hat{\mathbb{C}} - \hat{D}$ <u>for each</u> $t \in \mathbb{R}$, $w(\infty,t) \equiv 0$, <u>and</u> $w_t(z,t)$ <u>continuous in</u> $\hat{\mathbb{C}} - D \times \mathbb{R}$, $w(z,0) \equiv 0$, <u>and</u> $a = b = c = d = 0$ <u>in</u> $\hat{\mathbb{C}} - \hat{D}$, <u>then</u>

$$\frac{1}{2\pi i}\int_0^t \int_\Gamma \left[w_\tau(\zeta,\tau)\Omega_\tau^{(1)}(z,t;\zeta,\tau)d\zeta - \overline{w_\tau(\zeta,\tau)}\Omega_\tau^{(2)}(z,t;\zeta,\tau)d\overline{\zeta}\right]dt =$$

$$= \begin{cases} -w(z,t), & z \notin \hat{D} \\ 0 & , z \in D \end{cases}, t \in \mathbb{R}.$$

<u>Proof</u>. Let $z \in \mathbb{C} - \hat{D}$ be a point of $K_R: = \{\zeta: |\zeta| < R\}$ such that $2|z| < R$ and $\hat{D} \subset K_R$. The coefficients a, b, c, d vanish outside $\hat{D}$. w is a solution of (1) in $(K_R - \hat{D}) \times \mathbb{R}$. By theorem 1

$$w(z,t) = \frac{1}{2\pi i}\int_0^t \int_{\partial(K_R - \hat{D})} \Big[w_\tau(\zeta,\tau)\Omega_\tau^{(1)}(z,t;\zeta,\tau)d\zeta$$

$$- \overline{w_\tau(\zeta,\tau)}\quad \Omega_\tau^{(2)}(z,t;\zeta,\tau)d\overline{\zeta}\Big]d\tau .$$

As $w(z,t) = 0\left((z)^{-1}\right)$, $w_t(z,t) = 0\left((z)^{-1}\right)$ $(z \to \infty)$, it follows from (13) as R tends to infinity that the part of the integral taken over $|\zeta| = R$ tends to zero. If $z \in D$ the left side of the last equation has to be replaced by 0.

Theorem 3. A continuous solution of (1) in $\hat{D} \times \mathbb{R}$ with $w(z,0) \equiv 0$ where a, b, c, d vanish outside $\hat{D}$ may be represented in the form

$$(14) \quad w(z,t) = \frac{1}{2\pi i} \int_0^t \int_\Gamma \Big\{ \Phi_\tau(\zeta,\tau)\Omega_\tau^{(1)}(z,t;\zeta,\tau)d\zeta - \overline{\Phi_\tau(\zeta,\tau)}\Omega_\tau^{(2)}(z,t;\zeta,\tau)d\overline{\zeta} \Big\} d\tau \quad (z\varepsilon D,\ t\varepsilon\mathbb{R})$$

where

$$(15) \qquad \Phi(z,t) := \frac{1}{2\pi i} \int_\Gamma w(\zeta,\tau) \frac{d\zeta}{\zeta - z}$$

is an analytic function of z in D, continuous in $\hat{D}$, and a continuously differentiable function of t in $\mathbb{R}$.

Proof. The integral equation (2) now has the form

$$(16) \qquad w = \underset{\sim}{T}w + \Phi.$$

As $\underset{\sim}{T}w$ is an analytic function of z in $\hat{\mathbb{C}} - \hat{D}$, continuously differentiable with respect to t in $\mathbb{R}$, and

$$(\underset{\sim}{T}w)(\infty,t) \equiv 0 , \quad (\underset{\sim}{T}w)(z,0) \equiv 0$$

it follows by theorem 2 that

$$\frac{1}{2\pi i}\int_0^t \int_\Gamma \Big\{\frac{\partial}{\partial\tau}(\underset{\sim}{T}w)(\zeta,\tau)\Omega_\tau^{(1)}(z,t;\zeta,\tau)d\zeta -$$

$$- \overline{\frac{\partial}{\partial\tau}(\underset{\sim}{T}w)(\zeta,\tau)}\,\Omega_\tau^{(2)}(z,t;\zeta,\tau)d\overline{\zeta}\Big\}d\tau = 0 \quad (z\varepsilon D,\ t\varepsilon R)\ .$$

As w and $\underset{\sim}{T}w$ are continuous functions in $\hat{D} \times \mathbb{R}$, Φ is continuous there too and moreover Φ is continuously differentiable with respect to t and analytic in z. If one replaces w in the integral of (11) by the right hand side of (16) one gets (14) by using the last equality. (15) is to be found by the Cauchy formula for Φ in replacing Φ by $w - \underset{\sim}{T}w$ where $\underset{\sim}{T}w$ is observed to be analytic in $\hat{\mathbb{C}} - \hat{D}$ for every $t \,\varepsilon\, \mathbb{R}$ vanishing at infinity.

The formulas (14), (15) may be considered as a device for constructing all solutions of (1) with vanishing initial data. They now will be used to prove another representation for the solutions of (1).

<u>Theorem 4</u>. <u>Every solution of</u> (1) <u>with</u> $a = b = c = d = 0$ <u>outside</u> $\hat{D}$ <u>with</u> $w(z,0) \equiv 0$ <u>is representable by</u>

$$w(z,t) = \Phi(z,t) + \int_0^t \int_D \Big\{\Phi_\tau(\zeta,\tau)\Gamma_\tau^{(1)}(z,t;\zeta,\tau)$$

$$+ \overline{\Phi_\tau(\zeta,\tau)}\,\Gamma_\tau^{(2)}(z,t;\zeta,\tau)\Big\}d\xi d\eta d\tau\ ,$$

<u>where</u> Φ <u>is the function in</u> (15), <u>and</u> $\Gamma^{(k)}$ $(k = 1,2)$ <u>are given by</u>

$$\pi\Gamma^{(1)}(z,t;\zeta,\tau) := \Omega_{\overline{\zeta}}^{(1)}(z,t;\zeta,\tau),\quad \pi\Gamma^{(2)}(z,t;\zeta,\tau) := \Omega_{\zeta}^{(2)}(z,t;\zeta,\tau).$$

<u>Proof</u>. The theorem follows at once by an application of the Green identity for the domain $D\cap\{\zeta : |\zeta - z| > \varepsilon\}$, formula (14), and allowing ε tend to zero (see Vekua [9]).

As the kernels $\Omega^{(k)}$ as functions of (ζ,τ), with fixed (z,t), are solutions of the following equations

$$\Omega^{(1)}_{\zeta\tau}(z,t;\zeta,\tau) - a(\zeta)\Omega^{(1)}_{\tau}(z,t;\zeta,\tau) - \overline{b(\zeta)}\,\Omega^{(2)}_{\tau}(z,t;\zeta,\tau) +$$

$$+ c(\zeta)\Omega^{(1)}(z,t;\zeta,\tau) + \overline{d(\zeta)}\,\Omega^{(2)}(z,t;\zeta,\tau) = 0 ,$$

$$\Omega^{(2)}_{\zeta\tau}(z,t;\zeta,\tau) - \overline{a(\zeta)}\,\Omega^{(2)}_{\tau}(z,t;\zeta,\tau) - b(\zeta)\Omega^{(1)}_{\tau}(z,t;\zeta,\tau) +$$

$$+ \overline{c(\zeta)}\,\Omega^{(2)}(z,t;\zeta,\tau) + d(\zeta)\Omega^{(1)}(z,t;\zeta,\tau) = 0$$

(see Gilbert-Schneider [7]) one has

$$\pi\,\Gamma^{(1)}_{\tau}(z,t;\zeta,\tau) = a(\zeta)\Omega^{(1)}_{\tau}(z,t;\zeta,\tau) + \overline{b(\zeta)}\Omega^{(2)}_{\tau}(z,t;\zeta,\tau) -$$

$$- c(\zeta)\Omega^{(1)}(z,t;\zeta,\tau) - \overline{d(\zeta)}\Omega^{(2)}(z,t;\zeta,\tau) ,$$

$$\pi\,\Gamma^{(2)}_{\tau}(z,t;\zeta,\tau) = \overline{a(\zeta)}\Omega^{(2)}_{\tau}(z,t;\zeta,\tau) + b(\zeta)\Omega^{(1)}_{\tau}(z,t;\zeta,\tau) -$$

$$- \overline{c(\zeta)}\Omega^{(2)}(z,t;\zeta,\tau) - d(\zeta)\Omega^{(1)}(z,t;\zeta,\tau)$$

or by integrating

$$\pi\,\Gamma^{(1)}(z,t;\zeta,\tau) = a(\zeta)\Omega^{(1)}(z,t;\zeta,\tau) + \overline{b(\zeta)}\Omega^{(2)}(z,t;\zeta,\tau) +$$

$$+ c(\zeta)\int_{\tau}^{t}\Omega^{(1)}(z,t;\zeta,s)ds + \overline{d(\zeta)}\int_{\tau}^{t}\Omega^{(2)}(z,t;\zeta,s)ds ,$$

$$\pi\, \Gamma^{(2)}(z,t;\zeta,\tau) = \overline{a(\zeta)}\,\Omega^{(2)}(z,t;\zeta,\tau) + b(\zeta)\,\Omega^{(1)}(z,t;\zeta,\tau) +$$

$$+ \overline{c(\zeta)} \int_\tau^t \Omega^{(2)}(z,t;\zeta,s)\,ds + d(\zeta) \int_\tau^t \Omega^{(1)}(z,t;\zeta,s)\,ds \,.$$

III <u>A Second Integral Equation for Solutions</u>.

Our preceding considerations, as well as those in [7], are made under the restrictive assumption (5). Another similar approach can be done without (5) for all a, b, c, d in $L_{p,2}(\mathbb{C})$ $(2 < p)$. For this reason the fundamental kernels $\Omega^{(k)}(z,\zeta)\,(k=1,2)$ of the equation

$$w_{\bar z} + aw + b\bar w = 0 \,,$$

as given by Vekua (see [9], III 8), are used. From formula (13.19) of Chapter III in [9] it is obvious that special solutions of (1) satisfy the integral equation

$$(17)\quad w(z,t) - \frac{1}{\pi}\int_0^t \int_{\mathbb{C}} \Big[\Big(c(\zeta)w(\zeta,\tau) + d(\zeta)\overline{w(\zeta,\tau)}\Big)\Omega^{(1)}(z,\zeta) +$$

$$+ \Big(\overline{c(\zeta)}\,\overline{w(\zeta,\tau)} + \overline{d(\zeta)}\,w(\zeta,\tau)\Big)\Omega^{(2)}(z,\zeta)\Big]\,d\xi\, d\eta\, dt = \Phi(z,t)$$

where Φ is a solution of

$$\Phi_{\bar z t} + a\Phi_t + c\bar\Phi_t = 0 \,.$$

This can be seen by differentiating (17) with respect to t and $\bar z$ after splitting the integral over D into one over D_ε and one over $|\zeta - z| < \varepsilon$ as above. Again we are interested in bounded solutions of (1) in the sense that they are bounded in $\mathbb{C} \times I$ for every interval I of $\mathbb{R}$. If w is such a bounded solution and fulfills (17) for some Φ then Φ_t has to be a bounded solution of

$$\omega_{\bar{z}} + a\omega + b\bar{\omega} = 0$$

in $\mathbb{C}$ for each t in $\mathbb{R}$. If as before (F_o, G_o) is the generating pair of this equation every bounded solution in $\mathbb{C}$ has the form

$$\lambda F_o + \mu G_o$$

with real constants λ and μ. By this Φ has to be of the form

$$\Phi(z,t) = \lambda(t) F_o(z) + \mu(t) G_o(z)$$

with real differentiable functions λ and μ of t in $\mathbb{R}$.

(i) $\underline{\lambda = \mu = 0}$. If the integral operator in (17) is denoted by $\underset{\sim}{P}$ the problem

$$w - \underset{\sim}{P}w = 0 \tag{18}$$

has to be solved. To show that it has only the trivial solution one observes the bound

$$\frac{1}{\pi} \int\limits_{\mathbb{C}} \Big(|c|+|d|\Big)\Big(|\Omega^{(1)}(z,\zeta)|+|\Omega^{(2)}(z,\zeta)|\Big) d\xi d\eta \ \leq \kappa < \infty .$$

Let

$$||w||_\kappa := \sup_{\substack{z\in\mathbb{C}, \\ \kappa|t|\leq 1}} |w(z,t)| \ ;$$

then for $z \in \mathbb{C}$ and $\kappa|t| \leq 1$ it follows from (18) that

$$|w(z,t)| \leq \kappa||w||_\kappa |t| .$$

But this means that w vanishes identically in $z \in \mathbb{C}$ and $\kappa|t| \leq 1$. As above, because of autonomity of $\underset{\sim}{L}$, w must vanish identically in $\mathbb{C} \times \mathbb{R}$.

(ii) $\underline{\mu = 0}$. Now the solution of

$$w - \underset{\sim}{P}w = \lambda F_o$$

is sought, and this can be done by iteration, namely

$$w_o := \lambda F_o \;,\; w_k := w_o + \underset{\sim}{P}w_{k-1} (k\in\mathbb{N}),\; w := \lim_{k\to+\infty} w_k = \sum_{k=0}^{\infty} \underset{\sim}{P}^k w_o \;.$$

The function w is then uniquely defined when the convergence of the series has been shown.

$$\underset{\sim}{P}^k \lambda F_o = \lambda_k F_k (k\in \mathbb{N}_o), \lambda_0 := \lambda, \lambda_k(t) = \int_0^t \lambda_{k-1}(\tau)d\tau \quad (t\in\mathbb{R}), \; (k\in\mathbb{N}),$$

$$F_k(z) := \frac{1}{\pi} \int_{\mathbb{C}} \Big[\Big(c(\zeta)F_{k-1}(\zeta) + d(\zeta)\overline{F_{k-1}(\zeta)}\Big)\Omega^{(1)}(z,\zeta) +$$

$$+ \Big(\overline{c(\zeta)F_{k-1}(\zeta)} + \overline{d(\zeta)}F_{k-1}(\zeta)\Big)\Omega^{(2)}(z,\zeta)\Big] d\xi d\eta \quad (k\in\mathbb{N}) \;.$$

The convergence follows from the estimates,

$$||F_k|| \leq \kappa^k ||F_o|| \;, \quad ||\lambda_k||_t \leq \frac{|t|^k}{k!} ||\lambda_o||_t \quad (k\in\mathbb{N}_o)$$

where

$$||F_k|| = \sup_{z\in\mathbb{C}} |F_k(z)| \;, \quad ||\lambda_k||_t := \sup_{|\tau|\leq|t|} |\lambda_k(\tau)|$$

so that

$$(19) \qquad (\underset{\sim}{F}\lambda)(z,t) := \sum_{k=0}^{\infty} \lambda_k(t) F_k(z) \qquad (z\in\mathbb{C},\ t\in\mathbb{R})$$

is the solution. The functions λ_k and F_k have the properties

$$\lambda_k(0) = 0 \ (k\in\mathbb{N}), \quad F_k(\infty) = 0 \ (k\in\mathbb{N}), \quad F_o(\infty) = 1.$$

(iii) $\underline{\lambda = 0}$. As in (ii)

$$(20) \qquad (\underset{\sim}{G}\,\mu)(z,t) := \sum_{k=0}^{\infty} \mu_k(t) G_k(z) \qquad (z\in\mathbb{C},\ t\in\mathbb{R})$$

is the unique solution of

$$w - \underset{\sim}{P}w = \mu G_o$$

where μ_k and G_k are defined similarly as λ_k respectively F_k and

$$\mu_k(0) = 0\ , \quad G_k(\infty) = 0 \quad (k\in N)\ , \quad G_o(\infty) = i\ .$$

(iv) $\underline{\lambda \text{ and } \mu \text{ arbitrary}}$. The solution of the general problem,

$$w - \underset{\sim}{P}w = \lambda F_o + \mu G_o\ ,$$

is given by

$$w = \underset{\sim}{F}\lambda + \underset{\sim}{G}\mu.$$

The operators $\underset{\sim}{F}$ and $\underset{\sim}{G}$ are indeed the same as in (8) and (9) because the F_k from (19) are special solutions of the system (7) as the G_k from (20) are of the system (10). The difference of two solutions of the k-th equation of (7), or (10), is a bounded solution of

$$\omega_{\bar{z}} + a\omega + b\bar{\omega} = 0$$

which vanishes at infinity. But this only can be the zero solution. Because of these considerations we can now ignore the assumption (5); that is a and b, as well as the coefficients c and d, only have to be in $L_{p,2}(\mathbb{C})$.

(v) $\Phi(z,t) = f(z)t$. If (17) has the form

$$(w - \underset{\sim}{P}w)(z,t) = f(z)t \qquad (f\varepsilon L_{p,2}(\mathbb{C})\,(2 < p))$$

which for example is of interest when an inhomogeneous equation $\underset{\sim}{L}w = f$, f independent of t, has to be solved, the same calculations give the uniquely defined solution in the form

$$w(z,t) = \sum_{k=0}^{\infty} f_k(z)\,\frac{t^{k+1}}{(k+1)!}\,, \qquad f_o := f\,,$$

$$f_k(z) := \frac{1}{\pi}\int_{\mathbb{C}}\Big\{\Big[c(\zeta)f_{k-1}(\zeta) + d(\zeta)\overline{f_{k-1}(\zeta)}\Big]\Omega^{(1)}(z,\zeta) +$$

$$+ \Big[\overline{c(\zeta)f_{k-1}(\zeta)} + \overline{d(\zeta)}f_{k-1}(\zeta)\Big]\Omega^{(2)}(z,\zeta)\Big\}d\xi d\eta \quad (k\varepsilon N).$$

IV. Piecewise Continuous Solutions.

In the following the coefficients a, b, c, d are supposed to vanish identically outside the closure of a regular bounded domain D and to belong to $L_p(\hat{D})$ for some $p > 2$.

Lemma 3: Let $\rho(z,t)$ be a Hölder continuous function of z, and a C^1 function of t, for $(z,t) \in \Gamma \times I$, $\Gamma := \partial D$, I an interval (or open set) of $\mathbb{R}$. Let Γ be the union of a finite number of bounded, smooth, nonintersecting, closed curves $\Gamma_k (0 \leq k \leq m)$, such that $\mathbb{C} - \Gamma$ consists of one multiply connected bounded domain $D = D^+$ and simply connected domains $D_k^- (0 \leq k \leq m), D^- = \mathbb{C} - \hat{D}^+$ where D_o^- is unbounded. Then

$$\Phi(z,t) := \frac{1}{2\pi i} \int_\Gamma \rho(\zeta,t) \frac{d\zeta}{\zeta - z} \tag{21}$$

is analytic in D^+ as well as in D^- for each $t \in I$, continuously differentiable with respect to t for each $z \notin \Gamma$ satisfying

$$\Phi_t(z,t) = \frac{1}{2\pi i} \int_\Gamma \rho_t(\zeta,t) \frac{d\zeta}{\zeta - z}$$

and

$$\Phi^+(z,t) := \lim_{\substack{\zeta \to z \\ \zeta \in D^+}} \Phi(\zeta,t) = \Phi(z,t) + \frac{1}{2}\rho(z,t)$$

$$(z \in \Gamma,\ t \in I).$$

$$\Phi^-(z,t) := \lim_{\substack{\zeta \to z \\ \zeta \in D^-}} \Phi(\zeta,t) = \Phi(z,t) - \frac{1}{2}\rho(z,t)$$

Remark: Here (21) for $z \in \Gamma$ has to be understood in the Cauchy sense. The proof of this Plemelj-Sokhotski type formulae is modeled along the lines of the classical proof. In this regard see Gakhov [5], p.51 and p.64 and Muskhelishvili [8], p.128. Alternatively we may write

$$\Phi^+(z,t) - \Phi^-(z,t) = \rho(z,t),\ \Phi^+(z,t) + \Phi^-(z,t) = 2\Phi(z,t).$$

Lemma 4. Let ρ and Γ, D^+, D^- be as in the preceding lemma and $I = \mathbb{R}$, a, b, c, d in $L_p(\hat{D}^+)$ vanishing outside D^+, $\Omega^{(1)}$, $\Omega^{(2)}$ the fundamental kernels of the operator $\underset{\sim}{L}$ as given in Theorem 1. If then $\rho(z,0)$ vanishes identically on Γ

$$(22) \qquad w(z,t) = \frac{1}{2\pi i}\int_0^t \int_\Gamma \Big\{\rho_\tau(\zeta,\tau)\Omega_\tau^{(1)}(z,t;\ \zeta,\tau)d\zeta -$$

$$- \overline{\rho_\tau(\zeta,\tau)}\Omega_\tau^{(2)}(z,t;\zeta,\tau)d\overline{\zeta}\Big\}d\tau$$

is a solution of (1) in each component of $\mathbb{C} - \Gamma$, and in particular, of $w_{\bar{z}t} = 0$ in $\mathbb{C} - \hat{D}^+$. Furthermore, $w(z,0) \equiv 0$, and $w(z,t)$ fulfills the jump conditions,

$$(23) \qquad w^+(z,t) = w(z,t) + 0.5\rho(z,t) \qquad (z\varepsilon\Gamma,\quad t\varepsilon R).$$

$$w^-(z,t) = w(z,t) - 0.5\rho(z,t)$$

Remark: Here the first integral in (22) for $z \ \varepsilon \ \Gamma$ has to be understood in the Cauchy sense.

Proof. Because of the local behavior of the fundamental kernels (13) and the Plemelj-Sokhotski formulae for parameter dependent integrals it follows with (21)

$$(w-\Phi)^+(z,t) = (w-\Phi)(z,t) = (w-\Phi)^-(z,t) \qquad (z\varepsilon\Gamma,\ t\varepsilon\mathbb{R}).$$

Therefore $w - \Phi$ is continuous even on Γ so that (23) follows. That (22) is a solution of (1) follows by direct computation and the differential equation system for the kernels (see Gilbert-Schneider [7],(2.13)).

$$\Omega^{(1)}_{\bar{z}t} + a\Omega^{(1)}_{t} - b\overline{\Omega^{(2)}_{t}} + c\Omega^{(1)} - d\overline{\Omega^{(2)}} = 0 \quad ,$$

$$\Omega^{(2)}_{\bar{z}t} + a\Omega^{(2)}_{t} - b\overline{\Omega^{(1)}_{t}} + c\Omega^{(2)} - d\overline{\Omega^{(1)}} = 0 \quad .$$

Theorem 5: Let ρ have the properties stated in Lemma 4. Let v be a solution of the associated equation

$$(24) \qquad \underset{\sim}{L}^{*}v := \frac{\partial}{\partial t}\left(v_{\bar{z}} - av - \overline{bv}\right) + cv + \overline{dv} = 0 \ ,$$

defined in $\hat{D} \times \mathbb{R}$ and which vanishes at $t = T \in \mathbb{R}$ for all $z \in \mathbb{C}$. A necessary condition for ρ to represent the Hölder continuous boundary data w^{+} of a solution of (1) in D, which furthermore has identically vanishing initial data is that

$$(25) \qquad \operatorname{Im} \int_{0}^{T}\int_{\Gamma} \rho_{t}(\zeta,t)v_{t}(\zeta,t)\,d\zeta dt = 0 \ ,$$

and

$$(26) \qquad \int_{0}^{t}\int_{\Gamma} \left\{\rho_{\tau}(\zeta,\tau)\Omega^{(1)}_{\tau}(z,t;\zeta,\tau)d\zeta - \overline{\rho_{\tau}(\zeta,\tau)}\Omega^{(2)}_{\tau}(z,t;\zeta,\tau)d\bar{\zeta}\right\}d\tau$$

$$= 0 \quad (z \in D^{-} \ , \ t \in \mathbb{R}) \ .$$

Proof. Equation (24) is a conclusion of the identity of Green (see [7], (2.18)). To see the validity of (25) one has to consider

$$u(z,t):=\begin{cases} u_1(z,t)-w(z,t), & z\varepsilon D^+,\ t\varepsilon\mathbb{R} \\ u_1(z,t), & z\varepsilon D^-,\ t\varepsilon\mathbb{R}\ , \end{cases}$$

$$(27)\quad u_1(z,t):=\frac{1}{2\pi i}\int_0^t\int_\Gamma \Big\{\rho_\tau(\zeta,\tau)\Omega_\tau^{(1)}(z,t;\zeta,\tau)d\zeta -$$

$$-\overline{\rho_\tau(\zeta,\tau)\Omega_\tau^{(2)}(z,t;\zeta,\tau)d\zeta}\Big\}d\tau$$

which fulfills the boundary condition

$$u^+ - u^- = u_1^+ - u_1^- - w^+ = \rho - w^+$$

on $\Gamma \times \mathbb{R}$ and for each fixed t in $\mathbb{R}$

$$u(z,t) = O\left(|z|^{-1}\right) \quad (z \to \infty)\ .$$

If on $\Gamma \times \mathbb{R}$ $w^+ = \gamma$ then u is in $\mathbb{C} \times \mathbb{R}$ a continuous solution of (1), vanishing identically in t for $z = \infty$, and identically in z for $t = 0$. By the proof of Lemma 1, u vanishes identically.

<u>Theorem</u> <u>6</u>. <u>The</u> <u>condition</u> (25) <u>is</u> <u>sufficient</u> <u>for</u> ρ <u>to</u> <u>be</u> <u>the</u> <u>boundary</u> <u>values</u> <u>of</u> <u>a</u> <u>solution</u> <u>of</u> (1) <u>in</u> $D^+ \times \mathbb{R}$.

<u>Proof</u>. The function u, given in (27) is a solution of (1) in $D^+ \times \mathbb{R}$ and in $D^- \times \mathbb{R}$ and on $\Gamma \times \mathbb{R}$

$$u_1^+ - u_1^- = \rho.$$

Since (25) holds u_1 vanishes identically in $D^- \times \mathbb{R}$ so that u_1^- vanishes identically on $\Gamma \times \mathbb{R}$.

V. General Linear Riemann Boundary Value Problem.

Let Γ, a, b, c, d be as in the preceding section. Let g, h, γ be Hölder continuous functions of z on Γ where γ may also be a function of t which is continuously differentiable in that variable. Furthermore g is to have no zeros on Γ.

Theorem 7. Let w be a piecewise continuous solution of (1) in $\mathbb{C} \times \mathbb{R}$ with a jump over $\Gamma \times \mathbb{R}$; namely, on $\Gamma \times \mathbb{R}$

$$w^{+} = gw^{-} + \overline{hw^{-}} + \gamma \ . \tag{28}$$

Furthermore, let w have the homogeneous data

$$w(z,0) \equiv 0 (z\varepsilon\mathbb{C}),\ w(\infty,t) \equiv 0 \quad (t\varepsilon\mathbb{R}).$$

If v is any solution of the associated problem (24), such that $v(z,T) \equiv 0$ for some fixed T, and all $z \varepsilon \mathbb{C}$. Moreover, let

$$v^{-} = v^{+}g - \overline{v^{+}h}\ \overline{\frac{dz}{ds}} \Big/ \frac{dz}{ds} \tag{29}$$

on $\Gamma \times \mathbb{R}$, where s denotes the arc length parameter on Γ, then

$$\operatorname{Im} \int_0^T \int_\Gamma \left\{ w_t^-(z,t)v_t^-(z,t) + \gamma_t(z,t)v_t^+(z,t) \right\} dzdt = 0 \ . \tag{30}$$

Proof. From (28) it follows

$$\operatorname{Im} \int_0^T \int_\Gamma w_t^+(z,t)v_t^+(z,t)dzdt = \operatorname{Im} \int_0^T \int_\Gamma (gw_t^- v_t^+ dz - \overline{hw_t^- v_t^+} dz + \lambda_t v_t^+ dz)dt.$$

With (29) and the Green's identity (30) can be deduced.

In the homogeneous case where $\gamma \equiv 0$, the equalities

$$\mathrm{Im} \int_0^T \int_T w_t^+(z,t) v_t^+(z,t) dz dt = \mathrm{Im} \int_0^T \int_\Gamma w_t^-(z,t) v_t^-(z,t) dz dt = 0$$

hold instead.

VI. The Special Riemann Boundary Value Problem.

Let now Γ consist of a single contour not passing through the origin, and let h be identically zero so that the problem to be studied is,

$$(31) \qquad \underset{\sim}{L} w = 0 \quad \text{in} \quad \mathbb{C}-\Gamma \ , \quad w^+ = g w^- + \gamma \quad \text{on} \quad \Gamma \times \mathbb{R}.$$

As in the case of analytic functions $(a = b = c = d = 0$ in $\mathbb{C}$) the solution depends on the index of g, that is

$$\text{ind } g := \frac{1}{2\pi i} \int_\Gamma d \log g \ .$$

In the following the notation

$$w(z,t) := \begin{cases} w^+(z,t) & , \quad z \in D^+ \ , \quad t \in \mathbb{R} \\ w^-(z,t) & , \quad z \in D^- \ , \quad t \in \mathbb{R} \end{cases}$$

will be used. If $\text{ind } g = n (\in \mathbb{Z})$ then $z^{-n} g(z)$ is a Hölder continuous, nonvanishing complex function on Γ of index zero. Hence,

$$\phi(z) := \log g(z) - n \log z$$

is a single valued, Hölder continuous function on Γ.

$$\omega := w \exp(-\psi) \ , \quad \psi(z) := \frac{1}{2\pi i} \int_\Gamma \phi(\zeta) \frac{d\zeta}{\zeta - z}$$

leads (31) to the problem

$$(32)\quad \frac{\partial}{\partial t}\left(\omega_{\bar{z}} + a\omega + \tilde{b}\bar{\omega}\right) + c\omega + \tilde{d}\bar{\omega} = 0\ , \qquad \omega^{+} = z^{n}\omega^{-} + \tilde{\gamma}\ ,$$

$$\tilde{b} := b\exp(2i\operatorname{Im}\psi),\ \tilde{d} := d\exp(2i\operatorname{Im}\psi),$$

$$\tilde{\gamma} := \gamma\exp(-\psi - 0.5\phi).$$

(i) $n = \operatorname{ind} g = 0$. In the case of index zero the problem is

$$(33)\begin{cases} \frac{\partial}{\partial t}\left(\omega_{\bar{z}} + a\omega + \tilde{b}\bar{\omega}\right) + c\omega + \tilde{d}\omega = 0 & \text{in } (\mathbb{C}\setminus\Gamma) \times \mathbb{R}, \\ \omega^{+} = \omega^{-} + \tilde{\gamma} & \text{on } \Gamma \times \mathbb{R}\ . \end{cases}$$

If the fundamental kernels with respect to a, $\tilde{b}$, c, $\tilde{d}$ are denoted by $\tilde{\Omega}^{(k)}$ $(k=1,2)$ then

$$\omega(z,t) = (\underset{\sim}{I}\tilde{\gamma})(z,t) := \frac{1}{2\pi i}\int_{0}^{t}\int_{\Gamma}\Big\{\tilde{\gamma}_{\tau}(\zeta,\tau)\tilde{\Omega}_{\tau}^{(1)}(z,t;\zeta,\tau)d\zeta -$$

$$- \overline{\tilde{\gamma}_{\tau}(\zeta,\tau)\Omega_{\tau}^{(2)}(z,t;\zeta,\tau)d\bar{\zeta}}\Big\}d\tau$$

is the solution of (33) uniquely defined by

$$\omega(z,0) = 0 \qquad (z\in\mathbb{C}),\ \omega(z,t) = O(|z|^{-1}) \quad (z \to \infty,\ t \in \mathbb{R}).$$

This is a conclusion of the Plemelj-Sokhotzki formula (23). If to this solution a bounded solution $\omega_{o}(z,t)$ of the differential equation of (33) in $(\mathbb{C} \times \mathbb{R})$ with given data

$$\omega_{o}(z,0) \equiv 0\ , \quad \omega_{o}(\infty,t) \not\equiv 0 \quad (\omega_{o}(\infty,t) \in C^{1}(\mathbb{R}))$$

is added, one gets a solution of (33) with this "t-data" and vanishing "z-data". As it has been shown in I II this solution ω_o can be represented by the operators (19) (respectively (20)) associated with the differential equation in (33) as

$$\omega_o = \tilde{\underset{\sim}{F}}\lambda + \tilde{\underset{\sim}{G}}\mu \qquad (\omega_o(\infty,t) = \lambda(t) + i\mu(t)).$$

If now ω_1 is a solution of the differential equation in (33) with non-vanishing "z-data", namely

$$\omega_1(z,0) = \psi(z) \quad , \quad \omega_1(\infty,t) \equiv \psi(\infty) \quad ,$$

and u is a particular solution of

$$\frac{\partial}{\partial t}\left(u_{\bar{z}} + au + \tilde{b}\bar{u}\right) + cu + \tilde{d}u = c\psi + \tilde{d}\overline{\psi}$$

in the form (see III(v))

$$u(z,t) = \sum_{k=0}^{\infty} f_k(z)\,\frac{t^{k+1}}{(k+1)!}$$

with

$$f_o := c\psi+\tilde{d}\overline{\psi},\ f_k(z) := \frac{1}{\pi}\int_{\mathbb{C}}\left\{\left[c(\zeta)f_{k-1}(\zeta)+\tilde{d}(\zeta)\,\overline{f_{k-1}(\zeta)}\right]\tilde{\Omega}^{(1)}(z,\zeta)\ +\right.$$

$$\left. + \left[\overline{c(\zeta)}\,\overline{f_{k-1}(\zeta)} + \tilde{d}(\bar{\zeta})f_{k-1}(\zeta)\right]\tilde{\Omega}^{(2)}(\zeta,z)\right\}d\xi d\eta \qquad (k\in\mathbb{N})$$

then $\omega_1 - \psi + u$ is a solution of (33) with homogeneous data. Hence we have the following theorem:

<u>Theorem 8</u>. <u>The general solution of</u> (31) <u>in the case</u> $n = 0$ <u>is given by</u>

$$\left\{\underset{\sim}{I}\tilde{\gamma}(z,t) - \sum_{k=0}^{\infty} f_k(z)\,\frac{t^{k+1}}{(k+1)!} + \psi(z) + \left[\underset{\sim}{\tilde{F}}\lambda + \underset{\sim}{\tilde{G}}\mu\right](z,t)\right\} .$$

$$\cdot\ \exp\left[\frac{1}{2\pi i}\int_{\Gamma} \log g(\zeta)\frac{d\zeta}{\zeta - z}\right]$$

<u>where</u> $\lambda,\mu \in C^1(\mathbb{R}), \lambda(0) = \mu(0) = 0,\ \psi \in C(\mathbb{C}),\ \psi(\infty) = \lim_{z\to\infty}\psi(z) \in \mathbb{C}$.

(ii) <u>$n = \operatorname{ind} g > 0$</u> . A special solution of the inhomogeneous transformed problem (32) with

$$\omega(z,0) \equiv 0 \quad (z\in\mathbb{C}),\ \lim_{z\to\infty} z^{n+1}\omega(z,t) = \chi(t) \quad (t\in\mathbb{R}), \chi \in C^1(\mathbb{R})$$

can be found by the transformation

$$\text{(34)}\qquad \omega_1(z,t) := \begin{cases} \omega^+(z,t) \ , & z\in D^+ \\ z^n\omega^-(z,t), & z\in D^- \end{cases} ,\quad t\in\mathbb{R} .$$

This function has to be a solution of

$$\text{(35)}\quad \begin{cases} \dfrac{\partial}{\partial t}\left(\omega_{1\bar{z}} + a\omega_1 + b_1\bar{\omega}_1\right) + c\omega_1 + d_1\bar{\omega}_1 = 0 & \text{in } (\mathbb{C}-\Gamma)\times\mathbb{R}, \\ \omega_1^+ = \omega_1^- + \tilde{\gamma} & \text{on } \Gamma\times\mathbb{R} \end{cases}$$

where

$$b_1 := \begin{cases} \tilde{b} \ , & z\in D^+ \\ (z\bar{z}^{-1})^n\tilde{b} \ , & z\in D^- \end{cases} ,\quad d_1 := \begin{cases} \tilde{d} \ , & z\in D^+ \\ (z\bar{z}^{-1})^n\tilde{d} \ , & z\in D^- \end{cases} .$$

It must also satisfy

$$\omega_1(z,0) \equiv 0 \ (z\in\mathbb{C}),\quad \omega_1(z,t) = O(|z|^{-1}) \ (z\to\infty,\ t\in\mathbb{R}).$$

The solution of this problem is uniquely defined by

$$(36)\quad \omega_1(z,t = (\underset{\sim}{I}_1\tilde{\gamma})(z,t) := \frac{1}{2\pi i}\int_0^t\int_\Gamma \Big\{\tilde{\gamma}_\tau(\zeta,\tau)\Omega^{(1)}_{1\tau}(z,t;\zeta,\tau)d\zeta -$$

$$- \overline{\tilde{\gamma}_\tau(\zeta,\tau)}\Omega^{(2)}_{1\tau}(z,t;\zeta,\tau)d\overline{\zeta}\Big\}d\tau$$

where $\Omega_1^{(k)}$ $(k=1,2)$ are the fundamental kernels belonging to a, b_1, c, d_1. The uniqueness follows because the homogeneous problem (35) with $\tilde{\gamma} \equiv 0$ has only the trivial solution.

Using the function ω_1 we seek a special solution ω of (32) by means of the transformation (34), namely

$$(37)\quad \omega^+ := \omega_1^+ \text{ in } D^+ \times \mathbb{R}, \quad \omega^- := z^{-n}\omega_1^- \text{ in } D^- \times \mathbb{R}.$$

Now the general solution of the homogeneous problem

$$(38)\quad \begin{cases} \dfrac{\partial}{\partial t}\left[\omega_{\overline{z}} + a\omega + \tilde{b}\overline{\omega}\right] + c\omega + \tilde{d}\overline{\omega} = 0 & \text{in } (\mathbb{C}-\Gamma) \times \mathbb{R}, \\ \omega^+ - z^n\omega^- = 0 & \text{on } \Gamma \times \mathbb{R} \end{cases}$$

has to be found. In order that a solution ω behaves regularily at $z = \infty$ (for each t in $\mathbb{R}$), the solution ω_1 of the transformed problem (35) with $\tilde{\gamma} = 0$ may only have a pole of order no greater than n (for each real t). We have assumed that the coefficients of $\underset{\sim}{L}$ vanish near infinity in order that ω_1 be of the form

$$\alpha(z) + \beta(z,t) \quad , \quad \beta_{\overline{z}}(z,t) \equiv 0$$

in the vicinity of ∞ (for t in $\mathbb{R}$). As $\omega_1(z,0) \equiv 0$ the function α must be analytic.

Let $\hat{\underset{\sim}{F}}_k$, $\hat{\underset{\sim}{G}}_k$ be the operators (19)(respectively (20)) corresponding to the equation

$$(39) \qquad \frac{\partial}{\partial t}\left(w_{\bar{z}} + aw + (\bar{z}z^{-1})^k b_1 \bar{w}\right) + cw + (\bar{z}z^{-1})^k d_1 \bar{w} = 0 \quad (0 \le k \le n)$$

and λ, μ be real continuously differentiable functions of t in $\mathbb{R}$; then

$$w_o := \hat{\underset{\sim}{F}}_k \lambda + \hat{\underset{\sim}{G}}_k \mu$$

is a bounded solution of (39) with

$$w_o(\infty,t) = \lambda(t) + i\mu(t), \; w_o(z,0) = \hat{F}_k(z)\lambda(0) + \hat{G}_k(z)\mu(0),$$

where $\hat{F}_k, \hat{G}_k$ is the generating pair associated with $\left(a, (\bar{z}z^{-1})^k b_1\right)$. The operators $\underset{\sim}{F}_k, \underset{\sim}{G}_k$ defined by

$$\left(\underset{\sim}{F}_k \Phi\right)(z,t) := z^k \left(\hat{\underset{\sim}{F}}_k \Phi\right)(z,t) \quad , \quad \left(\underset{\sim}{G}_k \Phi\right)(z,t) := z^k \left(\hat{\underset{\sim}{G}}_k \Phi\right)(z,t)$$

are useful to construct solutions of (35) with $\tilde{\gamma} = 0$ having a pole of order k at infinity. The operators $\underset{\sim}{F}_k, \underset{\sim}{G}_k (0 \le k \le n)$ are linearly independent over $\mathbb{R}$. If more generally

$$\sum_{k=0}^{n} \left(\underset{\sim}{F}_k \lambda_k + \underset{\sim}{G}_k \mu_k\right) = 0$$

then

$$\sum_{k=0}^{n} z^{k-n} \left(\hat{\underset{\sim}{F}}_k \lambda_k + \hat{\underset{\sim}{G}}_k \mu_k\right)(z,t) \equiv 0 .$$

Allowing z to tend to infinity, one gets that

$$\lambda_n(t) + i\mu_n(t) \equiv 0 .$$

Proceeding in this manner it follows that

$$\lambda_k + i\mu_k = 0 \quad (0 \le k \le n).$$

On the other hand

$$\sum_{k=0}^{n} \left(\underset{\sim}{F}_k \lambda_k + \underset{\sim}{G}_k \mu_k\right)$$

is a solution of (35) with $\tilde{\gamma} = 0$ for every system λ_k, μ_k of real continuously differentiable functions of t in $\mathbb{R}$ having a pole of order less than or equal to n at infinity .

If in analogy to (37) $\underset{\sim}{\tilde{F}}_k, \underset{\sim}{\tilde{G}}_k$ $(0 \le k \le n)$ are defined by

$$\left(\underset{\sim}{\tilde{F}}_k \Phi\right)^+ := \left(\underset{\sim}{F}_k \Phi\right)^+ = z^k \underset{\sim}{\hat{F}}_k \Phi \quad , \quad \left(\underset{\sim}{\tilde{F}}_k \Phi\right)^- := z^{-n}\left(\underset{\sim}{F}_k \Phi\right)^- = z^{k-n} \underset{\sim}{\hat{F}}_k \Phi$$

$$\left(\underset{\sim}{\tilde{G}}_k \Phi\right)^+ := z^k \underset{\sim}{\hat{G}}_k \Phi \quad , \quad \left(\underset{\sim}{\tilde{G}}_k \Phi\right)^- := z^{k-n} \underset{\sim}{\hat{G}}_k \Phi ,$$

then

$$(40) \qquad \sum_{k=0}^{n} \left(\underset{\sim}{\tilde{F}}_k \lambda_k + \underset{\sim}{\tilde{G}}_k \mu_k\right) \quad \left(\lambda_k, \mu_k \in C^1(\mathbb{R}) \ , \ 0 \le k \le n\right)$$

is a solution of (38).

<u>Lemma 5</u>. <u>Every solution</u> $\tilde{\omega}$ <u>of problem</u> (38) <u>with</u> $\tilde{\omega}(z,0) \equiv 0$ <u>is given in the form</u> (40) <u>where</u> $\lambda_k(0) = \mu_k(0) = 0 (0 \le k \le n)$.

<u>Proof</u>. As $a, \tilde{b}, c, \tilde{d}$ vanish near infinity and $\tilde{\omega}(z,0) \equiv 0$, $\tilde{\omega}$ is analytic near infinity (for each t) so that

$$\tilde{\omega}(z,t) = 0\left(|z|^{k-n}\right) \quad (z \to \infty, \ t \in R, \ 0 \le k \le n) \ ,$$

or more precisely

$$\lim_{z\to\infty}\left[z^{n-k}\tilde{\omega}(z,t)\right] = \lambda_k(t) + i\mu_k(t) \qquad (t \in \mathbb{R}) ,$$

where λ_k and μ_k are in $C^1(\mathbb{R})$ and $\lambda_k(0) = \mu_k(0) = 0$.

We note, furthermore, that

$$\tilde{\omega} - \underset{\sim}{\tilde{F}}_k\lambda_k - \underset{\sim}{\tilde{G}}_k\mu_k$$

is a solution of (38), analytic outside $\hat{D}^+$. Moreover,

$$\lim_{z\to\infty} z^{n-k}\left[\tilde{\omega}-\underset{\sim}{\tilde{F}}_k\lambda_k-\underset{\sim}{\tilde{G}}_k\mu_k\right](z,t) = \lim_{z\to\infty}\left[z^{n-k}\tilde{\omega}-\underset{\sim}{\hat{F}}_k\lambda_k-\underset{\sim}{\hat{G}}_k\mu_k\right](z,t) =$$

$$= 0 \quad (t \in \mathbb{R}),$$

so that

$$\lim_{z\to\infty} z^{n-k+1}\left[\tilde{\omega}-\underset{\sim}{\tilde{F}}_k\lambda_k-\underset{\sim}{\tilde{G}}_k\mu_k\right](z,t) = \lambda_{k-1}(t) + i\mu_{k-1}(t) \qquad (t \in \mathbb{R}),$$

where λ_{k-1} and μ_{k-1} are $C^1(\mathbb{R})$ functions vanishing in $t = 0$.

By induction we arrive at a solution of (38) having the form

$$\omega := \tilde{\omega} - \sum_{\nu=0}^{k}\left(\underset{\sim}{\tilde{F}}_\nu\lambda_\nu + \underset{\sim}{\tilde{G}}_\nu\mu_\nu\right), \quad \lambda_\nu,\mu_\nu \in C^1(\mathbb{R}),$$

$$\lambda_\nu(0) = \mu_\nu(0) = 0 \qquad (0 \le \nu \le k)$$

with the properties

$$\omega(z,0) \equiv 0 \quad (z\in\mathbb{C}), \quad \omega(z,t) = 0\left(|z|^{-n-1}\right) \quad (z \to \infty,\ t \in \mathbb{R}).$$

But then

$$\omega_1 := \begin{cases} \omega^+ , & z \in D^+ \\ & , \quad t \in \mathbb{R} \\ z^n\omega^- , & z \in D^- \end{cases}$$

is a bounded solution in $\mathbb{C} \times \mathbb{R}$ of the homogeneous problem

(35) with $\tilde{\gamma} = 0$ and

$$\omega_1(z,0) \equiv 0 \quad (z \in \mathbb{C}), \quad \omega_1(z,t) = 0\left(|z|^{-1}\right) \quad (z \to \infty,\ t \in \mathbb{R})$$

so that ω_1 vanishes identically in $\mathbb{C} \times \mathbb{R}$. Adding to $\tilde{\omega}$ an arbitrary solution of (38) of the form (40), the assumption on $\tilde{\omega}(z,0)$ in the lemma appears unnecessary; consequently the following theorem holds.

<u>Theorem</u> <u>9</u>. <u>The</u> <u>general</u> <u>solution</u> <u>of</u> (31) <u>has</u> <u>the</u> <u>form</u>

$$w(z,t) = \left\{\omega(z,t) + \psi(z) + \sum_{k=0}^{n}\left(\tilde{\underset{\sim}{F}}_k\lambda_k + \tilde{\underset{\sim}{G}}_k\mu_k\right)(z,t)\right\}$$

$$\exp \frac{1}{2\pi i}\int_\Gamma \phi(\zeta)\frac{d\zeta}{\zeta - z}$$

<u>with</u>

$$w(z,t) := \begin{cases} \left(\underset{\sim}{I}_1\tilde{\gamma}\right)^+ - \sum\limits_{k=0}^{\infty} f_k(z)\,\dfrac{t^{k+1}}{(k+1)!} & \text{in } D^+ \\ z^{-n}\left(\underset{\sim}{I}_1\tilde{\gamma}\right)^- - \sum\limits_{k=0}^{\infty} z^{-n} f_k(z)\,\dfrac{t^{k+1}}{(k+1)!} & \text{in } D^- \end{cases}$$

<u>where</u> $\psi, \lambda_k, \mu_k\ (0 \le k \le n)$, $f_\nu\ (\nu \in \mathbb{N}_o)$ <u>are</u> <u>given</u> <u>by</u> <u>the</u> <u>initial</u> <u>data</u> <u>of</u> w, <u>and</u> <u>the</u> f_ν <u>are</u> <u>defined</u> <u>by</u> (42).

<u>Proof</u>. If $\tilde{\omega}_o$ is an arbitrary solution of (38) with given data

$$\psi(z) := \tilde{\omega}_o(z,0) \qquad (z \in \mathbb{C})$$

such that $\tilde{\omega}_o(z,t)$ does not have a singularity at $z = \infty$ (for all real t), and u is a special solution of the inhomogeneous problem

(41) $$\frac{\partial}{\partial t}\left(u_{\bar z} + au + \tilde{b}\overline{u}\right) + cu + \tilde{d}\overline{u} = \psi := c\psi + \tilde{d}\overline{\psi} \quad \text{in } (\mathbb{C}-\Gamma)\times\mathbb{R},$$

$$u^+ = z^n u^- \quad \text{on} \quad \Gamma \times \mathbb{R}, \quad u(z,0) \equiv 0 \quad \text{in} \quad \mathbb{C}, \; u(\infty,t) \equiv 0 \text{ in } \mathbb{R},$$

then

$$\tilde{\omega} := \tilde{\omega}_o - \psi + u$$

is a solution of (38) with homogeneous data

$$\tilde{\omega}(z,0) \equiv 0 \quad \text{in} \quad \mathbb{C} .$$

Hence, by lemma 5 $\tilde{\omega}$ may be given in the form (40), where

$$\lambda_k(0) = \mu_k(0) = 0 \quad (0 \le k \le n).$$

To find a special solution of the inhomogeneous problem (41), we consider the equation

$$\frac{\partial}{\partial t}\left(v_{\bar z} + av + b_1\overline{v}\right) + cv + d_1\overline{v} = f , \quad f = \begin{cases} \Psi & \text{in } D^+ \\ z^n\Psi & \text{in } D^- \end{cases} ,$$

where

b_1 and d_1 are as in (35) and v is determined similarly by u as ω_1 was in (34) by ω. One such solution can be found by solving the integral equation

$$v - \underset{\sim}{P}_1 v = \hat{f}\, t ,$$

where

$$(\underset{\sim}{P}_1 v)(z,t) := \frac{1}{\pi}\int_o^t \iint_{\mathbb{C}} \left\{\left[c(\zeta)v(\zeta,\tau) + d_1(\zeta)\overline{v(\zeta,\tau)}\right]\Omega_1^{(1)}(z,\zeta) + \left[\overline{c(\zeta)v(\zeta,\tau)} + \overline{d_1(\zeta)}v(\zeta,\tau)\right]\Omega_1^{(2)}(z,\zeta)\right\} d\xi d\eta d\tau$$

and

$$\hat{f}(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \left[f(\zeta)\Omega_1^{(1)}(z,\zeta) + \overline{f(\zeta)}\Omega_1^{(2)}(z,\zeta)\right] d\xi d\eta .$$

Here $\Omega_1^{(k)}$ (k=1,2) are the fundamental kernels of the equation

$$w_{\bar{z}} + aw + b_1\bar{w} = 0 .$$

As shown in III(v), the solution is given by

$$v(z,t) = \sum_{k=0}^{\infty} f_k(z) \frac{t^{k+1}}{(k+1)!} ,$$

(42) $$f_o := \hat{f} , \quad f_k(z) := \frac{1}{\pi} \int_{\mathbb{C}} \left\{ \left[c(\zeta) f_{k-1}(\zeta) + d_1(\zeta) \overline{f_{k-1}(\zeta)} \right] \Omega_1^{(1)}(z,\zeta) \right.$$

$$\left. + \left[\overline{c(\zeta)}\, \overline{f_{k-1}(\zeta)} + \overline{d_1(\zeta)} f_{k-1}(\zeta) \right] \Omega_1^{(2)}(z,\zeta) \right\} d\xi d\eta \quad (k \in \mathbb{N}),$$

where $f_k(\infty) = 0 \quad (k \in \mathbb{N}_o)$.

By the lemma 5 and this special solution the general solution of (31) may be found.

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One of the principal boundary value problems in analytic function theory is the so-called Riemann boundary value problem. The simplest version of the problem requires the finding of an analytic function ϕ in $C\setminus\Gamma$, where Γ is a closed smooth contour, and a prescribed Hölder continuous jump is prescribed for ϕ across Γ. The solution of this problem may be given in terms of a Cauchy integral. In generalized analytic, as well as

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generalized hyperanalytic function theory, a Cauchy-type representation exists, which suggest that the Riemann problem may be solved in a similar way.

Gilbert and Schneider investigated the pseudoparabolic equation

$$\underset{\sim}{L}w := \frac{\partial}{\partial t}\left[w_{\bar{z}} + aw + b\bar{w}\right] + cw + d\bar{w} = 0 \quad ,$$

where $a,b,c,d \in L_{p,2}(C)$, $2<p$. Integral representations reminescent of those occurring for generalized analytic functions were found to be valid. This permits the posing and solving of what we refer to as an initial-boundary value problem of the Riemann type.

In the present work several new representations for initial value problems are obtained. An iterative scheme is presented for solving the initial-boundary value problem. These results are of interest for investigating wave motion in anisotropic, nonhomogeneous elastic materials.